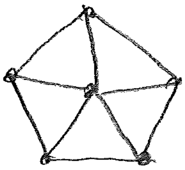
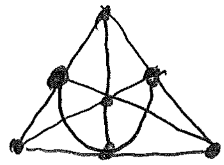
Def: k-uniform $k \geq 2$ hypergraph $H=(V,E)$
$V \equiv$ set of vertices $E \equiv$ set of edges
$\forall e \in E$ is subset of V of size k.

Ex: $k=2$ graphs



$k=3$ Fano plane



Remark: k-uniform hypergraph $\approx$ pure (homogeneous) simplicial complex

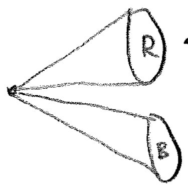
GOAL: hypergraphs are similar to graphs, yet many problems for hypergraph are exponentially harder. We illustrate this by surveying some open problems.

Ramsey th: **Def:** $r_k(s,t) \equiv \min N$ s.t. $\forall$ Red/blue coloring of ALL k-tuples on $[N]$ contains either red s-set or blue t-set. (Red s-set $\equiv$ ALL $\binom{s}{k}$ k tuples are red)

If $s=t=n$ $r_k(s,t) = r_k(n)$. If $k=2$ we omit it.

Q: How large are $r_k(s,t)$. (finite by Ramsey th)

Th (E-S, E)
$$2^{n/2} \leq r(n) \leq 2^{2n}$$



no red $s-1$ set

no blue $t-1$ set

- $r(s,t) \leq r(s-1,t) + r(s,t-1)$
- $N = 2^{n/2}$. Color edges R/B randomly

$$\Pr\left[\begin{matrix}\text{red/blue} \\ n-\text{set}\end{matrix}\right] \leq \binom{N}{n} \cdot 2 \cdot 2^{-\binom{n}{2}} \ll 1$$

Th (AKS, K)
$$r(3,t) = \Theta\left(\frac{t^2}{\log t}\right)$$

What about hypergraphs

$K=3$ Deleting one vertex gives
$$r_3(s,t) \leq r_2\left(r_3(s-1,t),\ r_3(s,t-1)\right)$$

Random coloring of triples shows that
$$r_3(n) \geq 2^{cn^2}$$

Random coloring gives no bound on $r_3(4,n)$

Th: (E-R)
$$r_3(s,t) \leq 2^{\binom{r(s-1,t-1)}{2}} \Rightarrow r_3(n) \leq 2^{2^{cn}}$$

Problem 1: \$500 show $r_3(n)$ is double exponential

Remark: $r_3(n,n,n,n)$ is! done by stepping up lemma.

Game: Players: Builder & Painter

Step $m+1$: new vertex v_{m+1} arrives. $\forall\, v_j$ $j \le m$
Builder decides if to add (v_j, v_{m+1}). Once new edge is in
Painter paints it R/B

$\tilde{r}(s,t) \equiv$ min # of edges of Builder which force a red s-clique or blue t-clique

Th (C-F-S) $\quad r_3(s,t) \le 2^{\tilde{r}(s,t)} \implies r_3(4,n) \le 2^{cn^2\log n}$

Lower Bound $\quad r_3(4,n) \ge 2^{cn\log n}$ (CFS)

Easier bound

Take random tournament on $N = 2^{cn}$ vert.

red edge  blue edge 

Problem: close gap $\quad 2^{cn\log n} \le r_3(4,n) \le 2^{cn^2\log n}$

Discrepancy: **Def**: set is ALMOST MONOCHROM $\equiv$ $(1-\varepsilon)$ fraction of k-tuples have the same color.

Facts:
- $\forall$ R/B colorings of edg. K_N has ALM. MON. set of size $c(\varepsilon)\log N$
- (C,F,S) $\forall$ R/B coloring of triples on $[N]$ has ALM. MONOCH. set of size $c(\varepsilon)\sqrt{\log N}$

Note: One would expect monochrom. set of size only $\log\log N$!

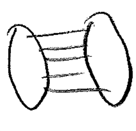
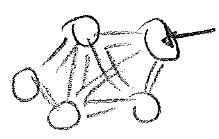
Problem: Prove $\forall$ R/B colorings of k-tuples on $[N]$ has ALM. MONOCHROM set of size $c(\varepsilon)\log^{\frac{1}{k-2}} N$

Remark one expects $\quad r_k(n) = \underbrace{2^{2^{\cdot^{\cdot^{2^{cn}}}}}}_{k-1}$ TOWER.

Turan problem

Def: H fixed k-uniform hypergraph $\quad ex(n,H) \equiv$ max # of edges in n-vertex k-uniform hypergraph not containing H.

Th (M, T)

- $ex(n, \triangle) = \frac{n^2}{4}$ 
- $ex(n, K_{r+1}) = \left(1-\frac{1}{r}\right)\frac{n^2}{2}$  $\frac{n}{r}$

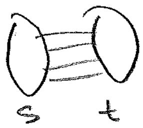
Th E-S: If $\chi(H) = r+1$ (define chromatic #) Then $ex(n,H) = \left(1-\frac{1}{r}\right)\frac{n^2}{2} + o(n^2)$

Min k s.t. vertices can be k-colored so that no edge is monochrom.

SOLUTION for ALL $r>1$

Th K-S-T, A-R-S: $ex(n, K_{s,t}) \geq cn^{2-1/s}$, Tight $t > (s-1)!$

$s \leq t$



in particular

$ex(n, K_{2,2}) = cn^{3/2}$

$ex(n, K_{3,3}) = cn^{5/3}$

construction: (n,d,λ) graphs

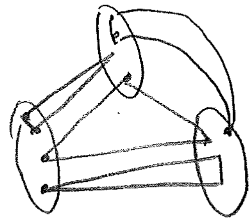
Define: Finite field, simple alg. geom

Hypergraphs

$k=3$ $K_4^3 \equiv 4$ vertices with ALL Triples

Q Turan: $ex(n, K_4^{(3)}) = ?$

Conj:



$n/3$

Def: $\pi(H) = \lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{k}}$

Conj: $\pi(K_4^3) = 5/9$ \$3000

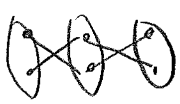
Difficult: exponentially many non-isomorphic extremal constructions.

Q: What is $\pi(K_{k+1}^{(k)})$?

Known: $\frac{1}{k} \leq 1 - \pi(K_{k+1}^{k}) \lesssim \frac{\log k}{2k}$

min density in k-graph with no indep set of size $k+1$

Q:



$K_{2,2,2}$

ALL Triples intersecting V_1, V_2, V_3, $|V_i| = 2$

what is $ex(n, K_{2,2,2}) \leq cn^{3-\frac{1}{2\cdot 2}}$

(maybe some construction of complexes with good eigenvalues might help)

Rusza-Szemeredi (Erdos-Brown-Sos)

Def: $f(n,p,q) = \max$ # of triples in 3-graph s.t. no p vertices span $\geq q$ triples.

$f(n,6,3) = ?$

$(6,3) \Rightarrow$ implies no

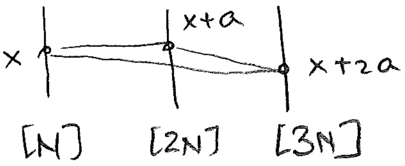
no

no

$\Rightarrow f(n,6,3) \leq \frac{1}{3}\binom{n}{2}$ every triple covers 2 pairs every pair covered once.

Th: R-S $\qquad f(n,6,3) = o(n^2)$. $\qquad$ Pf uses reduction to graphs and regularity lemma

R-S $\Rightarrow$ $\forall\, A \subseteq [N]$, $|A| = \delta N$ contains 3-term AP. $\qquad$ Roth Th



$\forall x \in [N]$, $a \in A$ $\equiv$ 3 AP free

This 3-graph is $(6,3)$ free

has $|A| \cdot N$ triples

Q: $\qquad f(n,7,4) = o(n^2)$? $\qquad$ Erdos-Frankl-Rodl

Note <u>no</u> codegree 3  so $f(n,7,4) \le \frac{2}{3}\binom{n}{2}$

Def: (i) cycle in graph $v_1 \dots v_k$ s.t. $(v_i, v_{i+1}) \equiv$ edge

(ii) cycle in hypergraph (Berge cycle) $v_1 \dots v_k$

vertices $v_1, \dots, v_k$ and edges $e_1, \dots, e_k$ s.t. $v_i \in e_i \cap e_{i+1}$ (addition mod k)

Remind Def $\chi(G) = \dots$!

(iii) girth $\equiv$ length of the shortest cycle.

REMARK: If Graph has large girth $\Rightarrow$ locally looks like tree so is 2-colorable locally. What about globally.

Th (E) $\exists$ k-uniform hypergraph with large girth and large chrom. #.

Sketch: $k=3$ girth $\equiv t$ Take edges rand. with prob.

$p = n^{-2+\frac{1}{t+1}}$, # of cycles of length $\le t$ $\approx n^{2t} p^t \ll n$ easy to destroy deleting few vertices

Union bound $\alpha(H) \approx n^{1-\frac{1}{2(t+1)}}$ polylog n $\Rightarrow \chi \gtrsim n^{\frac{1}{2(t+1)} - o(1)} = k$

Cor: $\exists$ 3-uniform hypergraph with girth t, $\chi = k$ and at most $k^{2(t+1)+o(1)}$ vertices

Q: find explicit construction

REMARK: in case of graphs one can get such constructions from (n,d,λ) graphs in particular for const d girth is logarithmic const. LPS.

Def: (n,d,λ) graph n - vertices d - regular all $|\lambda_i| \le \lambda$ $_{i \ge 2}$

Even simpler question

If $\chi(G) = s$ then $e(G) \ge \binom{s}{2}$ tight. need edge between every 2-color classes.

Tight $G = K_s$

Open problem: $f(k,s) = \min$ # of edges in k-uniform hypergraph with chromatic number s.

k = large, s - fixed $\sqrt{\frac{k}{\log k}}\, 2^k \le f(k,3) \le c k^2 2^k$ Minimal non 2-colorable hypergraph

Pf probabilistic: intresting to get explicit constructions.

k = fixed, s - large $f(k,s) \le \binom{(s-1)(k-1)+1}{k}$ complete k-graph on that many vert. has chrom # s.

Alon:

- can improve this by roughly $\binom{(s-1)k+1}{k} \cdot \frac{\log k}{k}$ using turan #
- lower bound $\gtrsim \left[\frac{(s-1)(k-1)}{k}\right]^k$ gap $\exp(k)$

Def: (-2) girth = min $t \geq 4$ such that H has t vertices spanning $t-2$ edges.
combinatorial girth

Def: Steiner triple system $\equiv$ 3-graph covering every pair of vertices once

STS exist if $2 \mid n-1$, $3 \mid \binom{n}{2}$.

Conj Erdos: $\exists$ STS with high (-2)-girth

easier conj Is there a 3 uniform hypergraph with cn^2 edges (c fixed) and arbitrarily high (-2) girth

Remark. $(6,3)$ condition can be thought of (-3) girth so then we have $o(n^2)$ edges.

Def. mod 2-cycle in H is a collection of edges that cover every vertex even # of times

Remark: cycles in graph has this property.

Claim. Let H be 4-uniform (4 is just for convinience) hypergraph on n vertices which has $\geq \binom{n}{2}$ edges. Then it has mod 2-cycle of order $\log n$.

Pf. Define G. vertices $\binom{n}{2}$ pairs

disjoint pairs adjacent if $(x,y,x_1,y_1) \equiv$ edge of H

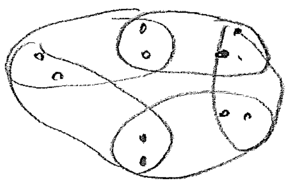
Note every edge of H gives 3 edges of G.

So $e(G) \geq 3\binom{n}{2} \Rightarrow G$ contains G' with min deg ≥ 3

G' has cycle of length $O(\log n)$

every level doubles compare to previous so this can continue only $O(\log n)$ steps

cycle in $G \equiv$ mod 2-cycle in H



Conj: suppose H has $\varepsilon\binom{n}{2}$ edges. Show still that it has mod 2 cycle of length polylog n.

Hamiltonicity

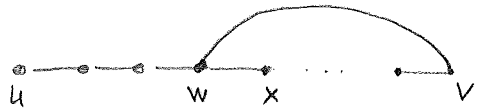
Def: Hamilton cycle is a cycle passing through every vertex of graph exactly once.

Q: When G is Hamiltonian (NP hard, suff. cond)

Expansion and HC

Posa: if $\forall\, X \subseteq G$, $|X| \leq k$: $|N(X)| \geq 2|X| \Rightarrow G$ has cycle of length $\geq 2k$

rotation



x new endpoint

Th K-S: Let G be (n,d,λ) graph with $\lambda \leq \frac{d}{\log n}$ then G is hamiltonian

Conj: $\lambda \leq \frac{d}{c}$, $c \equiv$ large constant is enough

Remark: since $\lambda \approx \sqrt{d}$ in best case works for $d \approx \log^2 n$ very sparse.

Def: H - 3 uniform hypergraph, n vert.

Tight HC ordering $v_1 \cdots v_n$ s.t. $(v_i, v_{i+1}, v_{i+2}) \equiv$ edge

loose HC

edges overlap in one vertex

Q: Find which spectral condition on H implies Tight/loose HC