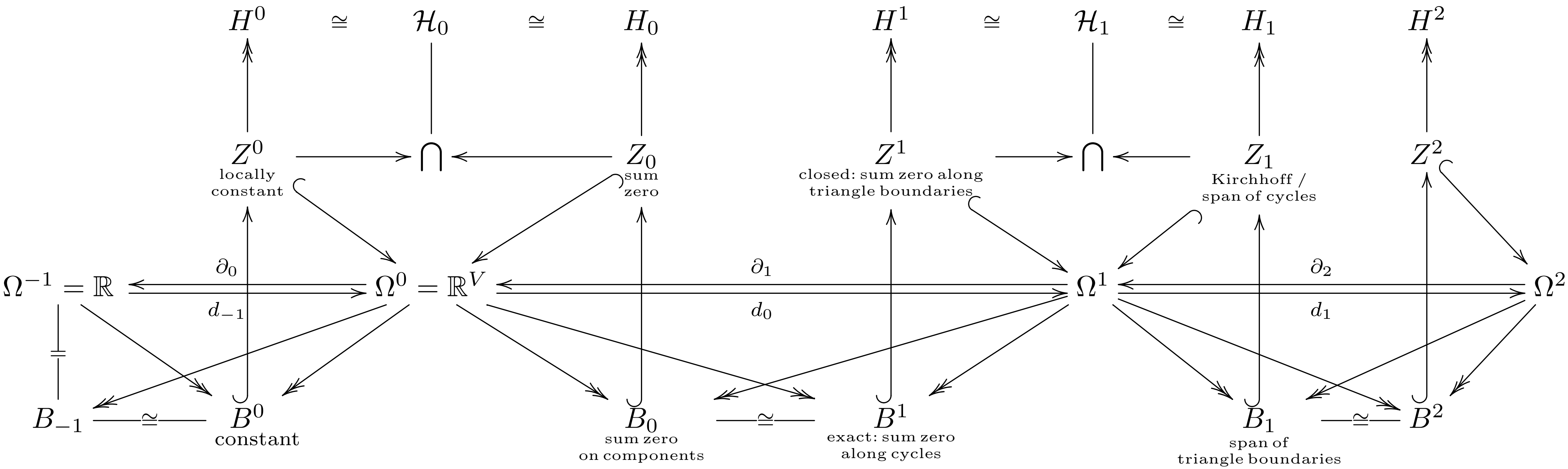
(Co)homology cheatsheet - Borel seminar, Les Diablerets

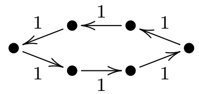
X - a simplicial complex
X^j - j-cells (cells with $j+1$ vertices).
$X^j_\pm$ - all oriented j-cells (orientations are orderings of the vertices, up to an even permutation)
$\Omega^j = \left\{ f : X^j_\pm \to \mathbb{R} \,\middle|\, f(\sigma \text{ "flipped"}) = -f(\sigma) \right\}$.
Boundary map: $\partial_j : \Omega^j \to \Omega^{j-1}$, $(\partial_j f)(\sigma) = \sum\limits_{v : v \cup \sigma \in X^j} f(v \cup \sigma)$
Coboundary (gradient) map: $d_j = \partial^*_{j+1} : \Omega^j \to \Omega^{j+1}$, $(d_j f)(\sigma) = \sum_{i=0}^{j+1} (-1)^i f(\sigma \backslash \sigma_i)$.

$Z_j = \ker \partial_j$	j-cycles	$Z^j = \ker d_j = B_j^\perp$	closed j-forms (or cocycles)
$B_j = \operatorname{im} \partial_{j+1}$	j-boundaries	$B^j = \operatorname{im} d_{j-1} = Z_j^\perp$	exact j-forms (or coboundaries)
$H_j = {}^{Z_j}/_{B_j}$	the j^{th} homology	$H^j = {}^{Z^j}/_{B^j}$	the j^{th} cohomology
$\mathcal{H}_j = H^j \cap H_j$	harmonic $j-$forms	$\Omega^j = B^j \oplus \mathcal{H}_j \oplus B_j$	(Eckmann-Hodge decomposition)



Exercises

(I will not assume anyone doing these for my next lectures, and I will repeat what we will need, but if you want to get better acquainted with the material now or in the future these are good ones to do).

1. For inner product spaces V, W and a linear map $T : V \to W$,
   (a) $(\ker T)^{\perp} = \operatorname{im} T^*$
   (b) $(\operatorname{im} T)^{\perp} = \ker T^*$
   (c) $\operatorname{im} TT^* = \operatorname{im} T$ (and $\ker TT^* = \ker T^*$)
2. Verify that the general forms for ∂_i, d_i coincide with the ones we defined for dimensions $0, 1, 2$.
3. Verify that $\partial_i = d_i^*$.
4. Show that $B_0 = \operatorname{im} \partial_1$ are the functions on the vertices which sum to zero on every connected component.
5. Deduce that $\dim H_0 = \dim H^0$ (the number of connected components minus one).
6. Show that the Kirchhoff forms ($Z_1 = \ker \partial_1$) are also the ones spanned by "cycles". By a "cycle" we mean a form which looks like  on some closed path and zero elsewhere (Hint at footnote[1]).
7. Show that $B^1 = \operatorname{im} d_0$ are the exact forms: those for which the sum over every closed cycle vanish (advice: use the previous exercise, or mimic the proof from complex analysis).
8. Show that in general $\dim H_i = \dim H^i$. (One possibility: use the Hodge decomposition).
   Warning: this is not true with a general ring replacing $\mathbb{R}$!
9. Compute $H^{0,1,2}$ and $H_{0,1,2}$. The more the merrier. Try a single triangle, the shell of a tetrahedron (icosahedron for the brave), an annulus, a Möbius strip, a torus, a Klein bottle...

[1] Pick a spanning tree, and look at cycles with one edge outside it. Add these cycles to an arbitrary Kirchhoff form to get one which is supported on the tree, and show that it must be zero.