Random Simplicial Complexes

Roy Meshulam
Technion – Israel Institute of Technology

Borel Seminar "High-Dimensional Expanders"
Les Diablerets, March 2015

Lecture I: Topology of the Binomial Model

Random Graphs

- The Erdős-Rényi model
- Topology of random graphs

The k-dimensional Erdős-Rényi model

- Homological connectivity
- Fundamental group
- The random k-tree process
- Collapsibility and top homology
- Embedding dimension

The Erdős-Rényi Model

$G(n, p)$ = random graphs on the $[n] = \{1, \ldots, n\}$
with independent edge probabilities p.

Ubiquity of Random Graphs

- Models for a multitude of natural phenomena, e.g. phase transition problems of statistical physics.
- A useful tool for showing the existence (indeed the prevalence) of graphs with favorable properties that are hard (or perhaps impossible) to produce explicitly.

A Classical Example: Ramsey Graphs

Erdős ('48): A typical $G \in G(n, \frac{1}{2})$ will contain neither a clique nor an independent set on $k = 2 \log_2 n$ vertices.

Remark: Current best explicit constructions achieve only
$k = 2^{(\log n)^\epsilon}$ (for any fixed $\epsilon > 0$).

Topology of a Random Graph

Theorem [Erdős-Rényi '58]:

For any function $\omega(n)$ that tends to infinity

$$\lim_{n\to\infty} \Pr\left[\ G \in G(n,p)\ :\ G \ \text{ connected}\ \right] =$$

$$\begin{cases} 0 & p = \frac{\log n - \omega(n)}{n} \\ 1 & p = \frac{\log n + \omega(n)}{n} \end{cases} \quad .$$

$$\lim_{n\to\infty} \Pr\left[\ G \in G(n,\frac{c}{n})\ :\ G \ \text{ acyclic}\ \right] =$$

$$\begin{cases} 0 & c > 1 \\ \sqrt{1-c}\cdot e^{\frac{2c+c^2}{4}} & c < 1 \end{cases} \quad .$$

A Model of Random Complexes

Y a simplicial complex , $Y^{(i)} = i$-dim skeleton of Y.
$Y(i) =$ oriented i-dim simplices of Y.
$f_i(Y) = |Y(i)|$.
$\Delta_{n-1} =$ the $(n-1)$-dimensional simplex on $V = [n]$.

$Y_k(n, p) =$ probability space of all complexes

$$\Delta_{n-1}^{(k-1)} \subset Y \subset \Delta_{n-1}^{(k)}$$

with probability distribution

$$\Pr(Y) = p^{f_k(Y)}(1-p)^{\binom{n}{k+1} - f_k(Y)} .$$

Homological Connectivity of Random Complexes

Fix $k \geq 1$ and a finite abelian group R.

Theorem [Linial-M '03 , M-Wallach '06]:

For any function $\omega(n)$ that tends to infinity

$$\lim_{n\to\infty} \Pr\left[\, Y \in Y_k(n,p) : \tilde{\mathrm{H}}_{k-1}(Y;R) = 0 \,\right] = \begin{cases} 0 & p = \frac{k\log n - \omega(n)}{n} \\ 1 & p = \frac{k \log n + \omega(n)}{n} \end{cases} .$$

Theorem [Hoffman-Kahle-Paquette '13]:

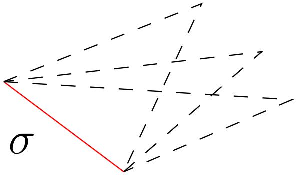
If $p > \frac{80k\log n}{n}$ then a.a.s. $\tilde{\mathrm{H}}_{k-1}(Y;\mathbb{Z}) = 0$.

Homological Connectivity - the Lower Bound

$\sigma \in \Delta_{n-1}(k-1)$ is isolated in $Y \in Y_k(n,p)$ if it is not contained in any $\tau \in Y(k)$.

$\sigma \in \Delta_{n-1}(k-1)$ isolated in Y

$\Rightarrow 0 \neq 1_\sigma \in \tilde{\mathrm{H}}^{k-1}(Y)$



If $p = \frac{k \log n - \omega(n)}{n}$ then

$$E[\text{number of isolated } \sigma' s] = \binom{n}{k}(1-p)^{n-k} \to \infty.$$

A second moment argument then implies that

$$\Pr[\tilde{\mathrm{H}}_{k-1}(Y;R) \neq 0] \to 1.$$

Fundamental Group of Random 2-Complexes

Theorem [Babson-Hoffman-Kahle '10]:

For any $\epsilon > 0$

$$\Pr \ [Y \in Y_2(n,p) : \pi_1(Y) = \{1\}] \to \begin{cases} 0 & p = n^{-\frac{1}{2}-\epsilon} \\ 1 & p = n^{-\frac{1}{2}+\epsilon} \end{cases} .$$

Theorem [M '11]:

If $\{G_n\}$ is a sequence of finite groups with $|G_n| \leq n^c$ then

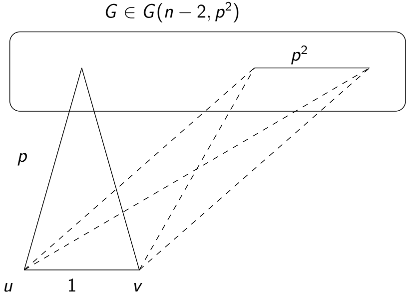
$$\Pr \ [Y \in Y_2(n, \frac{(3c+6)\log n}{n}) \ : \ \mathrm{Hom}(\pi_1(Y), G_n) = \{1\}] \to 1.$$

Simple Connectivity - the Upper Bound

For $Y \in Y_2(n, p)$ and $u \in [n]$ let $Y_u = \mathrm{St}_Y u$.

$Y_u \cap Y_v =$

$\mathrm{St}_Y uv \cup G$



It follows that if $p \geq \sqrt{\frac{10 \log n}{n}}$ then $Y_u \cap Y_v$ is connected for all u, v. Therefore, by the nerve lemma, $\pi_1(Y) = \{1\}$.

Hyperbolic Groups

Finitely presented groups

X - finite set, $F(X)$ - free group on X.
$R \subset F(X)$ - finite set of relations.
$G = \langle X | R \rangle = F(X)/(\text{normal subgroup generated by } R)$.

Area

Let $w \in F(X)$ such that $\overline{w} = 1 \in G$.
$\text{area}(w) =$ minimal n such that $w = (u_1^{-1} r_1^{\epsilon_1} u_1) \cdots (u_n^{-1} r_n^{\epsilon_n} u_n)$
for some $u_i \in F(X), r_i \in R, \epsilon_i \in \{\pm 1\}$.

Hyperbolicity

G is hyperbolic if there exist a constant A such that

$$\text{area}(w) \leq A \cdot \text{length}(w)$$

for all $w \in F(X)$ with $\overline{w} = 1$.

Examples

Hyperbolic groups

- Finite groups.
- Free groups.
- Fundamental groups of surfaces of genus ≥ 2:

$$G = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle.$$

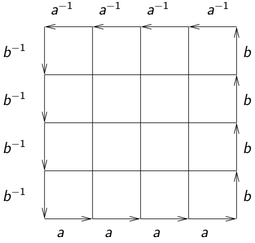
$\mathbb{Z}^2$ is not hyperbolic

$\mathbb{Z}^2 = \langle a, b | aba^{-1}b^{-1} \rangle$.

$w_n = a^n b^n a^{-n} b^{-n}$.

length$(w_n) = 4n$.

area$(w_n) = \Omega(n^2)$.



Isoperimetry and Hyperbolicity

X - a 2-dimensional simplicial complex.
γ - a simplicial null-homotopic loop in X of length $|\gamma|$.
$A_X(\gamma)$ - the minimal number of simplicies in a filling of γ.

The Isoperimetric Constant

$$\mathrm{I}(X) = \inf \left\{ \frac{|\gamma|}{A_X(\gamma)} : \gamma \sim 1 \right\}.$$

Theorem:

$$\mathrm{I}(X) > 0 \quad \Leftrightarrow \quad \pi_1(X) \ \text{ Hyperbolic } \ .$$

Local to Global Principle

Theorem [Gromov]:

Let $c > 0$ and let X be a 2-dimensional complex that satisfies:

$$\mathrm{I}(S) \geq c \text{ for all pure } S \subset X \text{ such that } f_2(S) \leq 10^6 c^{-2}.$$

Then:

$$\mathrm{I}(X) \geq 10^{-2} c.$$

The Structure of Sparse Complexes

A Density Invariant

$$\tilde{\mu}(X) = \min \left\{ \frac{f_0(Z)}{f_2(Z)} \ : \ Z \subset X \right\}.$$

Theorem [Babson-Hoffman-Kahle]:

- $\tilde{\mu}(X) > \frac{1}{2} \quad \Rightarrow \quad X$ is homotopic to a wedge of circles, 2-spheres and projective planes.
- For any $\epsilon > 0$ there exists a $c_\epsilon > 0$ such that

$$\tilde{\mu}(X) > \frac{1}{2} + \epsilon \ \ \Rightarrow \ \ \mathrm{I}(X) > c_\epsilon \ .$$

Simple Connectivity - the Lower Bound

Theorem [Babson-Hoffman-Kahle]:

$$p = o(n^{-\frac{1}{2}-\epsilon}) \quad \Rightarrow \quad \Pr[\,\mathrm{I}(Y) > 0\,] = 1 - o(1).$$

Sketch of Proof: Let S be a pure complex such that $f_2(S) \leq 10^6 c_\epsilon^{-2}$. If $\tilde{\mu}(S) \leq \frac{1}{2} + \epsilon$ then there exists a subcomplex $Z \subset S$ such that $f_0(Z) \leq (\frac{1}{2} + \epsilon) f_2(Z)$ and hence

$$\Pr[Y \supset S] \leq n^{f_0(Z)} p^{f_2(Z)} = n^{(\frac{1}{2}+\epsilon) f_2(Z)} \cdot o(n^{-\frac{1}{2}-\epsilon})^{f_2(Z)} = o(1).$$

Therefore Y a.a.s. satisfies the following condition:

$$S \subset Y \ \text{ and } \ f_2(S) \leq 10^6 c_\epsilon^{-2} \Rightarrow \tilde{\mu}(S) > \frac{1}{2} + \epsilon \Rightarrow \mathrm{I}(S) > c_\epsilon.$$

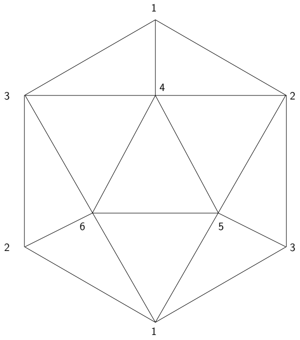
The local to global principle now implies that $\mathrm{I}(Y) > 10^{-2} c_\epsilon$.

Structure of $\pi_1(Y)$

Theorem [Costa-Farber '13]:

- If $p = n^{-\frac{3}{5}+\epsilon}$ then a.a.s. $\pi_1(Y)$ has 2-torsion.
- If $p = n^{-\frac{3}{5}-\epsilon}$ then a.a.s. $\pi_1(Y)$ is torsion free.

The 6-Point Projective Plane $\mathbb{P}^2$



If $p = n^{-\frac{f_0}{f_2}+\epsilon} = n^{-\frac{3}{5}+\epsilon}$ then a.a.s. $Y \supset Z \cong \mathbb{P}^2$ such that $\mathbb{Z}_2 = \pi_1(Z) \hookrightarrow \pi_1(Y)$.

Aspherical Subcomplexes in $Y_2(n, p)$

A 2-dimensional complex X is a aspherical if $\pi_2(X) = 0$.

Whitehead Conjecture

Let X be 2-dimensional complex. If X is aspherical and $Z \subset X$, then Z is aspherical.

Theorem [Costa-Farber '13]:

If $p = n^{-\frac{1}{2}-\epsilon}$ then a.a.s. Y satisfies the Whitehead conjecture, i.e. If $Y' \subset Y$ is aspherical then so are all $Y'' \subset Y'$.

An Easy Upper Bound on the Threshold for $\mathrm{H}_k(Y) = 0$

Let $z^{k-1}(Y) = \dim Z^{k-1}(Y)$, $\beta_k(Y) = \dim \mathrm{H}_k(Y)$.

By Euler-Poincaré:

$$\beta_k(Y) = f_k(Y) - \binom{n}{k} + z^{k-1}(Y) \ .$$

In particular, for $p = \frac{c}{n}$:

$$E[\beta_k] \ \geq \ E[f_k] - \binom{n-1}{k} \ \gtrsim \ (\frac{c}{k+1} - 1) \ \frac{n^k}{k!}.$$

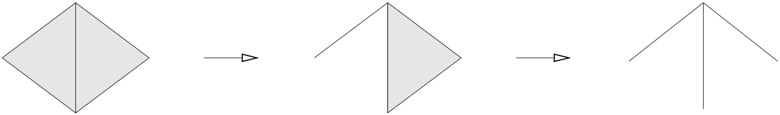
Hence

$$c > k+1 \quad \Rightarrow \quad \beta_k(Y) > 0 \ \text{ a.a.s.}$$

This can be (slightly) improved ...

k-Collapsibility

Suppose $\sigma \in X(k-1)$ is contained in a unique $\tau \in X(k)$.
An Elementary k-Collapse: $X \rightarrow X - \{\sigma, \tau\}$.



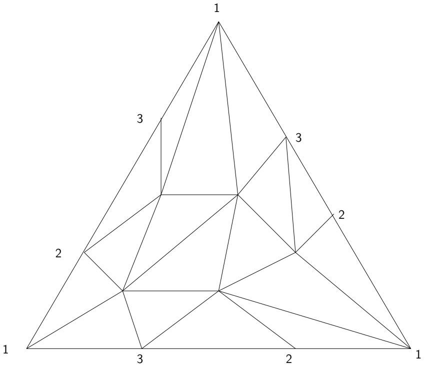
X is k-Collapsible if there is a sequence of k-collapses:

$$X = X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_m$$

such that $\dim X_m \leq k-1$.

$$X \ \ k - collapsible \ \ \overset{\Rightarrow}{\not\Leftarrow} \ \ X \ homotopic \ to \ \ a \ (k-1) - complex$$

The Dunce Hat



A contractible but non-collapsible complex

Boundaries of $(k+1)$-simplices

A Trivial Obstruction

If $Y \supset \partial\Delta_{k+1}$ then $H_k(Y) \neq 0$ and hence Y is not k-collapsible.

The Distribution of Boundaries

$$f(Y) = \text{number of } \partial\Delta_{k+1} \text{ in } Y \in Y_k(n, \tfrac{c}{n}).$$

$$E[f] = \binom{n}{k+2} p^{k+2} \sim \frac{c^{k+2}}{(k+2)!}.$$

$$f(Y) \sim \text{Poisson}(\lambda) \quad \text{with} \quad \lambda = \tfrac{c^{k+2}}{(k+2)!}.$$

The Threshold for k-Collapsibility

$x_k =$ minimal positive root of $\exp(-\frac{1-x}{kx}) = x$.
$\gamma_k = (kx_k(1-x_k)^{k-1})^{-1}$.
Remark: $\gamma_2 \doteq 2.455$, $\gamma_3 \doteq 3.089$, $\gamma_k = \Theta(\log k)$.

Theorem [Aronshtam-Linial-Łuczak-M]:

For any $c < \gamma_k$

$$\lim_{n\to\infty} \Pr\ [Y \in Y_k(n, \frac{c}{n}) :\ Y \ is\ k - collapsible] =$$

$$\lim_{n\to\infty} \Pr\ [Y \not\supset \partial\Delta_{k+1}] = \exp(-\frac{c^{k+2}}{(k+2)!}).$$

Theorem [Aronshtam-Linial]:

For any $c > \gamma_k$

$$\lim_{n\to\infty} \Pr\ [Y \in Y_k(n, \frac{c}{n}) :\ Y \ is\ k - collapsible] = 0.$$

The Threshold for $H_k(X) = 0$

$y_k =$ minimal positive root of $\exp(-\frac{(k+1)(1-y)}{1+ky}) = y$.
$\gamma_k^* = (k+1)(1+ky_k)^{-1}(1-y_k)^{-k}$.
Remark: $\gamma_2^* \doteq 2.753$, $\gamma_3^* \doteq 3.907$, $\gamma_k^* = k + 1 - \Theta(k^2 \exp(-k))$.

Theorem [Linial-Peled]:

For any $c < \gamma_k^*$

$$\lim_{n\to\infty} \Pr\ [Y \in Y_k(n, \frac{c}{n}) : H_k(Y; \mathbb{Q}) = 0] =$$

$$\lim_{n\to\infty} \Pr\ [Y \not\supset \partial\Delta_{k+1}] = \exp(-\frac{c^{k+2}}{(k+2)!}).$$

Theorem [Aronshtam-Linial]:

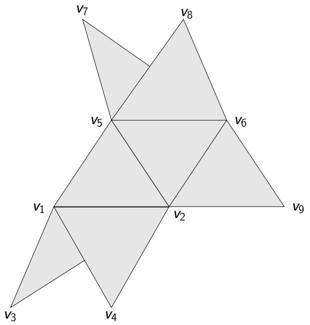
For any $c > \gamma_k^*$

$$\lim_{n\to\infty} \Pr\ [Y \in Y_k(n, \frac{c}{n}) : H_k(Y; \mathbb{Z}) = 0] = 0.$$

Rooted k-Trees

A k-Tree is a k-dimensional simplicial complex T on $V = \{v_1, \ldots, v_\ell\}$ such that for all $k+1 \leq i \leq \ell$ $\mathrm{link}(X[v_1, \ldots, v_{i-1}], v_i)$ is a $(k-1)$-simplex.
The Root of T is $\eta = \{v_1, \ldots, v_k\}$.

2-Tree



The Random k-Tree Process [ALLM]

Fix k and a parameter $\gamma > 0$. Inductively construct a chain of random k-trees rooted at $\eta = \{12 \ldots k\}$:

$$\{\eta\} = T_k(0, \gamma) \subset T_k(1, \gamma) \subset T_k(2, \gamma) \subset \cdots \quad .$$

Suppose $T_k(r, \gamma)$ is given. Independently for each $(k-1)$-face

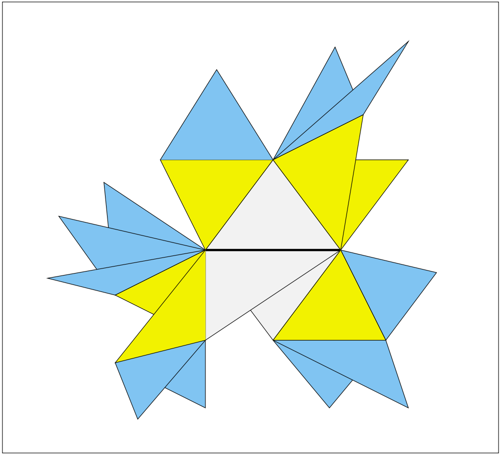
$$\tau \in T_k(r, \gamma) - T_k(r-1, \gamma)$$

pick $\mathrm{J}_\tau \sim \mathit{Poisson}(\gamma)$ new vertices $z_{\tau 1}, \ldots, z_{\tau \mathrm{J}_\tau}$ and let

$$T_k(r+1, \gamma) = T_k(r, \gamma) \cup \bigcup_\tau \{z_{\tau 1}\tau, \ldots, z_{\tau \mathrm{J}_\tau}\tau\}.$$

Fact: $T_k(r, \gamma)$ approximates the r-neighborhood of τ in $Y_k(n, \frac{\gamma}{n})$.

A 2-Tree Process



Pruning a Rooted k-Tree

Let T be a k-tree rooted at η.
$\{\tau_1, \ldots, \tau_t\}$ = Free $(k-1)$-faces of T different from η.
σ_i = the unique k-simplex containing τ_i.

A Pruning Step

$$T \to T - \{\tau_1, \sigma_1, \ldots, \tau_t, \sigma_t\} \ .$$

Main Point

Let γ_k = maximal such that for $\gamma < \gamma_k$

$$\lim_{r \to \infty} \Pr[\ T_k(r, \gamma) \searrow \eta \ \ in \ \ < r \ \ pruning \ steps\ ] = 1.$$

Then γ_k / n is the threshold for k-collapsibility.

Pruning of a Random k-Tree I

Probability of an Early Collapse

$\rho_k(r,\gamma) = \Pr[\ T_k(r,\gamma) \searrow \eta \ \ \textit{in} \ \ < r \ \ \textit{pruning steps}\ ].$

A Recursion for $\rho_k(r,\gamma)$

$\rho_k(1,\gamma) = \Pr[\ T_k(1,\gamma) = \eta\ ] = \exp(-\gamma)$.

$T_k(r,\gamma)$ collapses to η in $< r$ steps iff each k-simplex $\sigma \supset \eta$ contains a face $\eta' \neq \eta$ such that the random k-tree rooted in η' collapses to η' in $< r-1$ steps. Therefore:

$$\rho_k(r,\gamma) = \sum_{j=0}^{\infty} \Pr\ [\mathrm{J} = j](1-(1-\rho_k(r-1,\gamma))^k)^j =$$

$$\exp(-\gamma(1-\rho_k(r-1,\gamma))^k)\ .$$

Pruning of a Random k-Tree II

Let

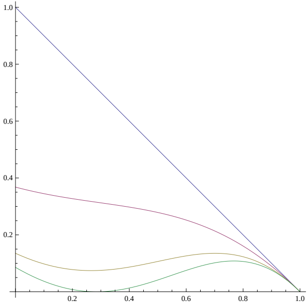
$$\rho_k(\gamma) = \lim_{r \to \infty} \rho_k(r, \gamma).$$

$\rho_k(\gamma)$ is the smallest positive root of the equation

$$u_k(\gamma, x) = \exp(-\gamma(1-x)^k) - x = 0.$$

Graphs of $u_2(\gamma, x)$ for

$\gamma = 0, 1, 2, 2.455$



Pruning of a Random k-Tree III

If $\gamma \geq 0$ is small then $\rho_k(\gamma) = 1$. Let

$$\gamma_k = \inf\{\gamma > 0 : \rho_k(\gamma) < 1\} \quad \textit{and} \quad x_k = \rho_k(\gamma_k).$$

The pair (γ_k, x_k) satisfies

$$u_k(\gamma, x) = \frac{\partial u_k}{\partial x}(\gamma, x) = 0.$$

It follows that x_k is the unique solution of

$$\exp(-\frac{1-x}{kx}) = x$$

and

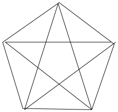
$$\gamma_k = (kx_k(1-x_k)^{k-1})^{-1}.$$

Embedding Dimension

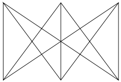
Any k-dimensional complex can be embedded in $\mathbb{R}^{2k+1}$.

Examples of k-dimensional complexes not embeddable in $\mathbb{R}^{2k}$:

The k-skeleton of the $(2k+2)$-simplex.
The $(k+1)$-fold join of 3-point spaces.



K_5



$K_{3,3}$

Theorem [Wagner]:

There exist constants $c_k < C_k$ such that:

- $Y \in Y_k(n, \frac{c_k}{n})$ a.a.s. embeds in $\mathbb{R}^{2k}$.
- $Y \in Y_k(n, \frac{C_k}{n})$ a.a.s. does not embed in $\mathbb{R}^{2k}$.

Lecture II: Cohomological Expansion

Expansion in Graphs and Complexes

- Cheeger constant of a graph
- Cohomological expansion of complexes
- Application to random complexes
- Expansion and topological overlap

Randomized construction of expanding complexes

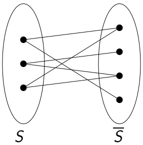
- Expander graphs and complexes
- A bounded degree model
- Spectral gaps and 2-Expansion
- Random LS-Complexes are 2-Expanders

The Graphical Cheeger Constant

Edge Cuts

For a graph $G = (V, E)$ and $S \subset V$, $\overline{S} = V - S$ let

$$e(S, \overline{S}) = |\{e \in E : |e \cap S| = 1\}|.$$



Cheeger Constant

$$h(G) = \min_{0<|S|\leq\frac{|V|}{2}} \frac{e(S, \overline{S})}{|S|}.$$

Spectral Gap

Laplacian Matrix

$G = (V, E)$ a graph, $|V| = n$.

The Laplacian of G is the $V \times V$ matrix L_G:

$$L_G(u, v) = \begin{cases} \deg(u) & u = v \\ -1 & uv \in E \\ 0 & \text{otherwise.} \end{cases}$$

Eigenvalues of L_G

$$0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G).$$

$$\mu_2(G) = \text{Spectral Gap of } G.$$

Expansion and Spectral Gap

Theorem [Alon-Milman, Tanner]:

For all $\emptyset \neq S \subsetneq V$

$$e(S, \overline{S}) \geq \frac{|S||\overline{S}|}{n} \mu_2.$$

In particular

$$h(G) \geq \frac{\mu_2}{2}.$$

Theorem [Alon, Dodziuk]:

If G is d-regular then

$$h(G) \leq \sqrt{2d\mu_2}.$$

Expanders can thus be defined using the spectral gap.

Simplicial Cohomology

X a simplicial complex on V, R a fixed abelian group.
i-face of $\sigma = [v_0, \cdots, v_k]$ is $\sigma_i = [v_0, \cdots, \widehat{v_i}, \cdots, v_k]$.
$C^k(X) =$ k-cochains = skew-symmetric maps $\phi : X(k) \to R$.
Coboundary Operator $d_k : C^k(X) \to C^{k+1}(X)$ given by

$$d_k\phi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i) \quad .$$

$d_{-1} : C^{-1}(X) = R \to C^0(X)$ given by
$d_{-1}a(v) = a$ for $a \in R$, $v \in V$.
$Z^k(X) =$ k-cocycles $= \ker(d_k)$.
$B^k(X) =$ k-coboundaries $= \mathrm{Im}(d_{k-1})$.
k-th reduced cohomology group of X:

$$\tilde{\mathrm{H}}^k(X) = \tilde{\mathrm{H}}^k(X; R) = Z^k(X)/B^k(X) \quad .$$

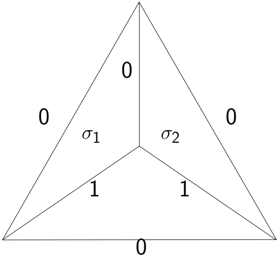
Cut of a Cochain

Cut determined by a k-cochain $\phi \in C^k(X; R)$:

$$\mathrm{supp}(d_k\phi) = \{\tau \in X(k+1) \ : \ d_k\phi(\tau) \neq 0\} \ .$$

Cut Size of ϕ: $\|d_k\phi\| = |\mathrm{supp}(d_k\phi)|$.

Example:



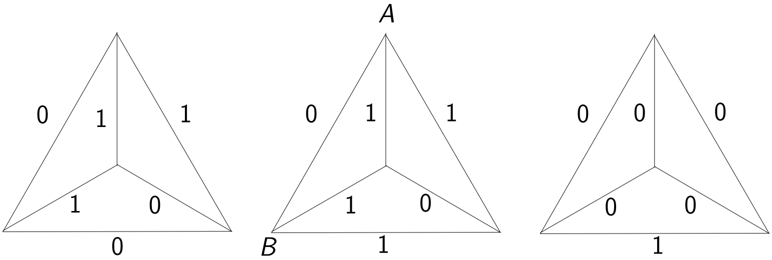
$$\|d_1\phi\| = |\{\sigma_1, \sigma_2\}| = 2$$

Hamming Weight of a Cochain

The Weight of a k-cochain $\phi \in C^k(X;R)$:

$$\|[\phi]\| = \min\ \{\ |\mathrm{supp}(\phi + d_{k-1}\psi)|\ :\ \psi \in C^{k-1}(X;R)\ \}.$$

Example: $\|\phi\| = 3$ but $\|[\phi]\| = 1$



ϕ $\qquad$ $d_0 1_{A,B}$ $\qquad$ $\phi - d_0 1_{A,B}$

Expansion of a Complex

Expansion of a Cochain

The expansion of $\phi \in C^k(X;R) - B^k(X;R)$ is

$$\frac{\|d_k\phi\|}{\|[\phi]\|}.$$

k-expansion Constant

$$h_k(X;R) = \min\left\{\frac{\|d_k\phi\|}{\|[\phi]\|} : \phi \in C^k(X;R) - B^k(X;R)\right\}.$$

Remarks:

- $h_k(X;R) > 0 \Leftrightarrow \tilde{H}^k(X;R) = 0$.
- In the sequel: $h_k(X) = h_k(X;\mathbb{F}_2)$.

Expansion of a Simplex I

Δ_{n-1} = the $(n-1)$-dimensional simplex on $V = [n]$.

Claim [M-Wallach, Gromov]:

$$h_{k-1}(\Delta_{n-1}) = \frac{n}{k+1}.$$

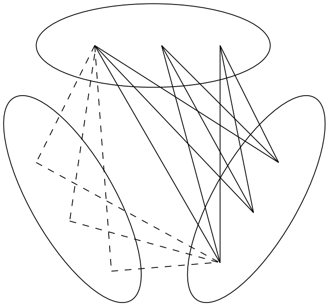
Example:

$[n] = \cup_{i=0}^{k} V_i \ , \ |V_i| = \frac{n}{k+1}$

$\phi = 1_{V_0 \times \cdots \times V_{k-1}}$

$\|[\phi]\| = (\frac{n}{k+1})^k$

$\|d_{k-1}\phi\| = (\frac{n}{k+1})^{k+1}$



Expansion of a Simplex II

Let $\phi \in C^{k-1}(\Delta_{n-1})$. For $u \in V$ define $\phi_u \in C^{k-2}(\Delta_{n-1})$ by

$$\phi_u(\tau) = \begin{cases} \phi(u\tau) & u \notin \tau \\ 0 & u \in \tau \end{cases}.$$

Let $\sigma \in \Delta_{n-1}(k-1)$ and $u \in V$. Then

$$d_{k-1}\phi(u\sigma) = \phi(\sigma) - \sum_{w \in \sigma} \phi(u(\sigma - w)) = \phi(\sigma) - d_{k-2}\phi_u(\sigma)$$

Therefore

$$\begin{aligned}
&(k+1)\|d_{k-1}\phi\| \\
&\qquad = |\{(\tau, u) \in \Delta_{n-1}(k) \times V : u \in \tau \in \operatorname{supp}(d_{k-1}\phi)\}| \\
&\qquad = |\{(\sigma, u) \in \Delta_{n-1}(k-1) \times V : \sigma \in \operatorname{supp}(\phi - d_{k-2}\phi_u)\}| \\
&\qquad = \sum_{u \in V} |\operatorname{supp}(\phi - d_{k-2}\phi_u)| \geq n\|[\phi]\|.
\end{aligned}$$

Homological Connectivity of Random Complexes

Fix $k \geq 1$ and a finite abelian group R.

Theorem [Linial-M '03 , M-Wallach '06]:

For any function $\omega(n)$ that tends to infinity

$$\lim_{n\to\infty} \Pr\left[\ Y \in Y_k(n,p) : \tilde{\mathrm{H}}_{k-1}(Y;R) = 0\ \right] = \begin{cases} 0 & p = \frac{k\log n - \omega(n)}{n} \\ 1 & p = \frac{k\log n + \omega(n)}{n} \end{cases} .$$

The Relevance of Expansion:

If $0 \neq [\phi] \in \tilde{\mathrm{H}}^{k-1}(\Delta_{n-1}^{(k-1)})$ then

$$\Pr\ \left[\ [\phi] \in \tilde{\mathrm{H}}^{k-1}(Y;\mathbb{F}_2)\ \right] = (1-p)^{\|d_k\phi\|}$$
$$\leq (1-p)^{\frac{n\|[\phi]\|}{k+1}}.$$

Expansion and Homological Connectivity

A weak threshold:

If $p = \frac{(k^2+k+1)\log n}{n}$ then a.a.s. $H^{k-1}(Y;\mathbb{F}_2) = 0$.

Proof:

$$\Pr\ [\ \tilde{\mathrm{H}}^{k-1}(Y;\mathbb{F}_2) \neq 0\ ] \leq \sum_{0 \neq [\phi] \in \tilde{\mathrm{H}}^{k-1}(\Delta_{n-1}^{(k-1)})} (1-p)^{\|d_{k-1}\phi\|} \leq$$

$$\sum_{m \geq 1} \binom{\binom{n}{k}}{m} (1-p)^{\frac{nm}{k+1}} \leq$$

$$\sum_{m \geq 1} (n^k n^{-\frac{k^2+k+1}{k+1}})^m =$$

$$\sum_{m \geq 1} (n^{-\frac{1}{k+1}})^m \to 0.$$

Weighted Expansion

X - n-dimensional pure simplicial complex.
A probability distribution on $X(k)$:

$$w(\sigma) = \frac{|\{\eta \in X(n) : \sigma \subset \eta\}|}{\binom{n+1}{k+1} f_n(X)}.$$

For $\phi \in C^k(X)$ let

$$\|\phi\|_w = \sum_{\{\sigma \in X(k) : \phi(\sigma) \neq 0\}} w(\sigma)$$

$$\|[\phi]\|_w = \min\{\|\phi + d_{k-1}\psi\| : \psi \in C^{k-1}(X)\}.$$

Weighted k-th Expansion:

$$\underline{h}_k(X) = \min\left\{\frac{\|d_k\phi\|_w}{\|[\phi]\|_w} : \phi \in C^k(X) - B^k(X)\right\}.$$

The Affine Overlap Property

Number of Intersecting Simplices

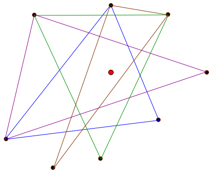
For $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}^k$ and $p \in \mathbb{R}^k$ let

$$\gamma_A(p) = |\{\sigma \subset [n] : |\sigma| = k+1 \ , \ p \in \mathrm{conv}\{a_i\}_{i \in \sigma}\}|.$$

Theorem [Bárány]:

There exists $p \in \mathbb{R}^k$ such that

$$f_A(p) \geq \frac{1}{(k+1)^k} \binom{n}{k+1} - O(n^k).$$



The Topological Overlap Property

Number of Intersecting Images

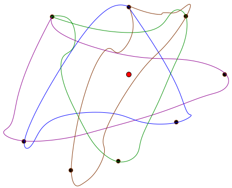
For a continuous map $f : \Delta_{n-1} \to \mathbb{R}^k$ and $p \in \mathbb{R}^k$ let

$$\gamma_f(p) = |\{\sigma \in \Delta_{n-1}(k) : p \in f(\sigma)\}|.$$

Theorem [Gromov]:

There exists $p \in \mathbb{R}^k$ such that

$$\gamma_f(p) \geq \frac{2k}{(k+1)!(k+1)} \binom{n}{k+1} - O(n^k).$$



Topological Overlap and Expansion

Number of Intersecting Images

For a continuous map $f : X \to \mathbb{R}^k$ and $p \in \mathbb{R}^k$ let

$$\gamma_f(p) = |\{\sigma \in X(k) : p \in f(\sigma)\}|.$$

Expansion Condition on X

Suppose that for all $0 \leq i \leq k-1$

$$\underline{h}_i(X) \geq \epsilon.$$

Theorem [Gromov]

There exists a $\delta = \delta(k, \epsilon)$ such that for any continuous map $f : X \to \mathbb{R}^k$ there exists a $p \in \mathbb{R}^k$ such that

$$\gamma_f(p) \geq \delta f_k(X).$$

Symmetric Matroids

Matroid:

An n-dimensional simplicial complex $M \subset 2^V$ such that $M[S]$ is pure for all $S \subset V$.

Homology of matroids:

$\tilde{H}_i(M) = 0$ for all $0 \leq i \leq \dim M - 1$.

Symmetric matroid:

$G = Aut(M)$ is transitive on the maximal faces.

Examples of Symmetric Matroids

The Partition Matroid $X_{n,m}$

Let $V_1, \ldots, V_{n+1}$ be $n+1$ disjoint sets, $|V_i| = m$.

$$X_{n,m} = \{\sigma \subset \bigcup_{i=1}^{n+1} V_i : \forall i \ \ |\sigma \cap V_i| \le 1\}.$$

Independence Matroid of Affine Space

$$\mathrm{IN}(\mathbb{F}_q^n) = \{\sigma \subset \mathbb{F}_q^n : \sigma \textit{ is linearly independent}\}.$$

Hermitian Unital with 65 Points

Independence matroid of the curve

$$H = \{[x, y, z] \in PG(2, 16) : x^5 + y^5 + z^5 = 0\}.$$

Expansion of Symmetric Matroids

Proposition [Lubotzky-M-Mozes]:

M symmetric matroid $\Rightarrow$ $\underline{h}_k(M) \geq 8^{-\dim M} \quad \forall\, k \leq \dim M - 1.$

Example: The Partition Matroid $X_{n,m}$

For $0 \leq k \leq n-1$

$$\underline{h}_k(X_{n,m}) \geq \frac{\binom{n+1}{k+1}}{\sum_{j=0}^{k+1} \left(\frac{2(m-1)}{m}\right)^j \binom{n-j}{n-k-1}}.$$

Corollary [Dotterrer-Kahle]:

$$\underline{h}_k(\text{Octahedral } n-\text{sphere}) = \underline{h}_k(X_{n,2}) \geq 1$$

and

$$\underline{h}_{n-1}(X_{n,m}) \geq \frac{n+1}{2^{n+1}-1}.$$

Expansion of Spherical Buildings

The Building $A_{n+1}(\mathbb{F}_q)$

Vertices: All nontrivial linear subspaces $0 \neq V \subsetneq \mathbb{F}_q^{n+2}$.
Simplices: $V_0 \subset \cdots \subset V_k$.

Theorem [Solomon, Tits]:

$$\dim \tilde{\mathrm{H}}_i(A_{n+1}(\mathbb{F}_q)) = \begin{cases} q^{\binom{n+2}{2}} & i = n \\ 0 & else. \end{cases}$$

Proposition [Gromov, LMM]:

$$\underline{h}_{n-1}(A_{n+1}(\mathbb{F}_q)) \geq \frac{1}{(n+2)!}.$$

Expander Graphs

(d, ϵ)-Expanders

A family of graphs $\{G_n = (V_n, E_n)\}_n$ with $|V_n| \to \infty$
with two seemingly contradicting properties:

- High Connectivity: $h(G_n) \geq \epsilon$.
- Sparsity: $\max_v \deg_{G_n}(v) \leq d$.

Pinsker:

Random $3 \leq d$-regular graphs are (d, ϵ)-expanders.

Margulis:

Explicit construction of expanders.

Lubotzky-Phillips-Sarnak, Margulis:

Ramanujan Graphs - an "optimal" family of expanders.

The Ubiquity of Expanding Graphs

Uses of Expanders

- Construction of efficient communication networks.
- Randomization reduction in probabilistic algorithms.
- Construction of good error correcting (LDPC) codes.
- Tools in computational complexity lower bounds.

Interactions with Other Areas

- Expansion and Kazhdan's property T.
- Expanders as spaces of maximal Euclidean distortion.
- Dimension expanders and representation theory.
- Expanders on finite simple groups.

Expander Complexes

Degree of a Simplex

For $\sigma \in X(k-1)$ let $\deg(\sigma) = |\{\tau \in X(k) : \sigma \subset \tau\}|$.
$D_{k-1}(X) = \max_{\sigma \in X(k-1)} \deg(\sigma)$.

(k, d, ϵ)-Expanders

A family of Complexes $\{X_n\}_n$ with $f_0(X_n) \to \infty$ such that

$$D_{k-1}(X_n) \leq d \quad \text{and} \quad h_{k-1}(X_n) \geq \epsilon.$$

Random Complexes as Expanders

$Y \in Y_k(n, p = \frac{k^2 \log n}{n})$ is a.a.s. a $(k, \log n, 1)$-expander.

Problem

Do there exist (k, d, ϵ)-expanders with $k \geq 2$ and fixed d, ϵ ?

Latin Squares

Definitions

$\mathbb{S}_n$ = Symmetric group on $[n]$.
$(\pi_1, \ldots, \pi_k) \in \mathbb{S}_n^k$ is legal if $\pi_i(\ell) \neq \pi_j(\ell)$ for all ℓ and $i \neq j$.
A Latin Square is a legal n-tuple $L = (\pi_1, \ldots, \pi_n) \in \mathbb{S}_n^n$.
$\mathcal{L}_n$ = Latin squares of order n with uniform measure.

The Usual Picture

$L = (\pi_1, \ldots, \pi_n) \leftrightarrow T_L \in M_{n \times n}([n])$
$T_L(i, \pi_k(i)) = k$ for $1 \leq i, k \leq n$.

Example for $n = 4$

$\pi = (1234)$

$L = (Id, \pi, \pi^2, \pi^3)$

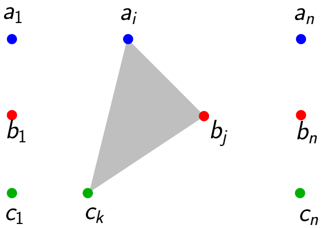
$T_L =$

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

The Complete 3-Partite Complex

$$V_1 = \{a_1, \ldots, a_n\}\ ,\ V_2 = \{b_1, \ldots, b_n\}\ ,\ V_3 = \{c_1, \ldots, c_n\}$$

$$T_n = V_1 * V_2 * V_3 = \{\sigma \subset V : |\sigma \cap V_i| \leq 1 \text{ for } 1 \leq i \leq 3\}$$



$$T_n \simeq S^2 \vee \cdots \vee S^2 \qquad (n-1)^3 \text{ times}$$

Latin Square Complexes

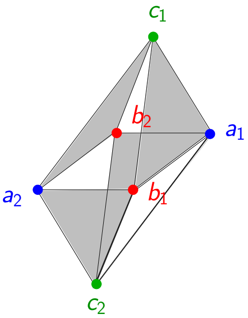
$L = (\pi_1, \ldots, \pi_n) \in \mathcal{L}_n$ defines a complex $Y(L) \subset T_n$ by

$$Y(L)(2) = \left\{ \ [a_i, b_j, c_{\pi_i(j)}] \quad : \quad 1 \leq i, j \leq n \right\}.$$

Example: $n = 2$

$$L = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array}$$

$Y(L) =$



$$Y\left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array} \right) \cup Y\left(\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 2 \\ \hline \end{array} \right) = T_2$$

Random Latin Squares Complexes

Multiple Latin Squares

For $\underline{L}^d = (L_1, \ldots, L_d) \in \mathcal{L}_n^d$ let $Y(\underline{L}^d) = \cup_{i=1}^d Y(L_i)$.

The Probability Space $\mathcal{Y}(n, d)$

$\mathcal{L}_n^d$ = d-tuples of Latin squares of order n with uniform measure.
$\mathcal{Y}(n, d) = \{Y(\underline{L}^d) : \underline{L}^d \in \mathcal{L}_n^d\}$ with induced measure from $\mathcal{L}_n^d$.

Theorem [Lubotzky-M]:

There exist $\epsilon > 0, d < \infty$ such that

$$\lim_{n\to\infty} \Pr\left[Y \in \mathcal{Y}(n, d) : h_1(Y) > \epsilon\right] = 1.$$

Remark: $\epsilon = 10^{-11}$ and $d = 10^{11}$ will do.

Idea of Proof

Fix $0 < c < 1$ and let $\phi \in C^1(T_n; \mathbb{F}_2)$.

$$\phi \text{ is } \begin{cases} c - \text{small} & \text{if } \|[\phi]\| \leq cn^2 \\ c - \text{large} & \text{if } \|[\phi]\| \geq cn^2 \end{cases}$$

c-Small Cochains

Lower bound on expansion in terms of the spectral gap of the vertex links.

c-Large Cochains

Expansion is obtained by means of a new large deviations bound for the probability space $\mathcal{L}_n$ of Latin squares.

2-Expansion and Spectral Gap

Notation

For a complex $T_n^{(1)} \subset Y \subset T_n$ let:
$Y_v = \mathrm{lk}(Y, v) =$ the link of $v \in V$.
$\mu_v =$ spectral gap of the $n \times n$ bipartite graph Y_v.
$\tilde{\mu} = \min_{v \in V} \mu_v$.
$d = D_1(Y) =$ maximum edge degree in Y.

Theorem [LM]:

If $\|[\phi]\| \leq cn^2$ then

$$\|d_1\phi\| \geq \left(\frac{(1 - c^{1/3})\tilde{\mu}}{2} - \frac{d}{3} \right) \|[\phi]\|.$$

Spectral Gap of Random Graphs

Random Bipartite Graphs

$\tilde{\pi} = (\pi_1, \ldots, \pi_d) \in \mathbb{S}_n^d$ defines a graph $G = G(\tilde{\pi})$ by

$$E(G) = \{ \ (i, \pi_j(i)) \ : \ 1 \leq i \leq n, 1 \leq j \leq d \ \} \subset [n]^2.$$

$\mathcal{G}(n, d)$= uniform probability space $\{G(\tilde{\pi}) : \tilde{\pi} \in \mathbb{S}_n^d\}$.

Theorem [Friedman]:

For a fixed $d \geq 100$:

$$\Pr[G \in \mathcal{G}(n, d) \ : \ \mu_2(G) > d - 3\sqrt{d}] = 1 - O(n^{-2}).$$

Expansion of c-Small Cochains

Links as Random Graphs

Let $Y = Y(\underline{L}^d)$ be a random complex in $\mathcal{Y}(n,d)$.
Then $Y_v = \mathrm{lk}(Y,v)$ is a random graph in $\mathcal{G}(n,d)$.
Therefore

$$\Pr[\ \tilde{\mu} \geq d - 3\sqrt{d}\ ] = 1 - O(n^{-1}).$$

Corollary:

Let $d \geq 100$ and $c < 10^{-3}$. If $\|[\phi]\| \leq cn^2$ then

$$\begin{aligned}\frac{\|d_1\phi\|}{\|[\phi]\|} &\geq \frac{(1-c^{1/3})\tilde{\mu}}{2} - \frac{d}{3} \\ &\geq \frac{(1-c^{1/3})(d-3\sqrt{d})}{2} - \frac{d}{3} > 1.\end{aligned}$$

Large Deviations for Latin Squares

The Random Variable $f_{\mathcal{E}}$

$\mathcal{E}$ - a family of 2-simplices of T_n, $|\mathcal{E}| \geq cn^3$.
For a Latin square $L \in \mathcal{L}_n$ let

$$f_{\mathcal{E}}(L) = |Y(L) \cap \mathcal{E}|.$$

Then

$$E[f_{\mathcal{E}}] = \frac{|\mathcal{E}|}{n} \geq cn^2.$$

Theorem [LM]:

For all $n \geq n_0(c)$

$$\Pr[L \in \mathcal{L}_n \ : \ f_{\mathcal{E}}(L) < 10^{-3}c^2n^2] < e^{-10^{-3}c^2n^2}.$$

Topological Overlap Property for $\mathcal{Y}(n, d)$

Corollary:

There exist $\delta > 0$ and d such that $Y \in \mathcal{Y}(n, d)$ a.a.s. satisfies the following:

For any continuous map $f : Y \to \mathbb{R}^2$ there exists $p \in \mathbb{R}^2$ such that

$$\gamma_Y(p) \geq \delta n^2.$$

Lecture III: Garland's Method

Eigenvalues and Cohomology

- Higher Laplacians
- Simplicial Hodge theorem
- Garland's Theorem

Applications of Garland's method

- Cohomology of discrete groups
- Topology of random flag complexes
- Hypergraph matching and homology

Random Surfaces

- The Harer-Zagier model
- The genus of a random surface
- Related models

The Graph Laplacian

$G = (V, E)$ a graph

The Laplacian of G is the $V \times V$ matrix L_G:

$$L_G(u, v) = \begin{cases} \deg(u) & u = v \\ -1 & uv \in E \\ 0 & \text{otherwise} \end{cases}$$

Spectrum of L_G: $\quad 0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$

The Spectral Gap

$\lambda_2(G)$ controls the expansion of G and the convergence rate of a random walk on G. In particular, $\lambda_2(G) > 0 \Leftrightarrow G$ connected.

Higher Laplacians

A positive weight function $c(\sigma)$ on the simplices of X induces an Inner product on $C^k(X) = C^k(X;\mathbb{R})$:

$$(\phi, \psi) = \sum_{\sigma \in X(k)} c(\sigma)\phi(\sigma)\psi(\sigma) \quad .$$

Adjoint $\quad d_k^* : C^{k+1}(X) \to C^k(X)$

$$(d_k \phi, \psi) = (\phi, d_k^* \psi) \ .$$

$$C^{k-1}(X) \underset{d_{k-1}^*}{\overset{d_{k-1}}{\rightleftarrows}} C^k(X) \underset{d_k^*}{\overset{d_k}{\rightleftarrows}} C^{k+1}(X)$$

The reduced k-Laplacian of X is the positive semidefinite operator

$$\Delta_k = d_{k-1} d_{k-1}^* + d_k^* d_k : C^k(X) \to C^k(X) \quad .$$

Harmonic Cochains

The space of Harmonic k-cochains

$$\ker \Delta_k = \{\phi \in C^k(X) : d_k\phi = 0 \ , \ d_{k-1}^*\phi = 0\}.$$

Simplicial Hodge Theorem:

$$C^k(X) = \mathrm{Im}\, d_{k-1} \oplus \ker \Delta_k \oplus \mathrm{Im}\, d_k^* \ .$$

$$\ker \Delta_k \cong \tilde{\mathrm{H}}^k(X;\mathbb{R}).$$

$\mu_k(X) =$ minimal eigenvalue of Δ_k.

A Vanishing Criterion:

$$\mu_k(X) > 0 \Leftrightarrow \tilde{\mathrm{H}}_k(X;\mathbb{R}) = 0.$$

Eigenvalues and Cohomology

Let X be a pure d-dimensional complex with weight function:

$$c(\sigma) = (d - \dim \sigma)!|\{\tau \in X(d) : \tau \supset \sigma\}|.$$

For $\tau \in X$ consider the link $X_\tau = \mathrm{lk}(X, \tau)$ with a weight function given by $c_\tau(\alpha) = c(\tau\alpha)$.

Theorem [Garland '72]:

Let $0 \leq \ell < k < d$. Then:

$$\min_{\tau \in X(\ell)} \mu_{k-\ell-1}(X_\tau) > \frac{\ell+1}{k+1} \quad \Rightarrow \quad H^k(X; \mathbb{R}) = 0.$$

In particular:

$$\min_{\tau \in X(d-2)} \mu_0(X_\tau) > \frac{d-1}{d} \quad \Rightarrow \quad H^{d-1}(X; \mathbb{R}) = 0.$$

Sketch of Proof I

For $\phi \in C^k(X)$ define $\phi_\tau \in C^{k-\ell-1}(X_\tau)$ by $\phi_\tau(\alpha) = \phi(\tau\alpha)$.

Garland's Identity

$$\binom{k}{\ell+1}(\Delta_k\phi, \phi) = \sum_{\tau \in X(\ell)} (\Delta_{k-\ell-1}\phi_\tau, \phi_\tau) - \binom{k}{\ell}\|\phi\|^2.$$

Proof of Garland's Theorem:

Suppose that

$$\min_{\tau \in X(\ell)} \mu_{k-\ell-1}(X_\tau) > \frac{\ell+1}{k+1}$$

and let $0 \neq \phi \in C^k(X)$ such that $\Delta_k\phi = \mu_k(X)\phi$.

Sketch of Proof II

By Garland's identity:

$$\begin{aligned}
\mu_k(X)\binom{k}{\ell+1}\|\phi\|^2 &= \binom{k}{\ell+1}(\Delta_k\phi,\phi) \\
&= \sum_{\tau\in X(\ell)} (\Delta_{k-\ell-1}\phi_\tau,\phi_\tau) - \binom{k}{\ell}\|\phi\|^2 \\
&\geq \min_{\tau\in X(\ell)} \mu_{k-\ell-1}(X_\tau) \sum_{\tau\in X(\ell)} \|\phi_\tau\|^2 - \binom{k}{\ell}\|\phi\|^2 \\
&\geq \left(\min_{\tau\in X(\ell)} \mu_{k-\ell-1}(X_\tau)\binom{k+1}{\ell+1} - \binom{k}{\ell}\right)\|\phi\|^2 \\
&> \left(\frac{\ell+1}{k+1}\binom{k+1}{\ell+1} - \binom{k}{\ell}\right)\|\phi\|^2 = 0.
\end{aligned}$$

Complexes with Expanding Links

The Projective Plane Graph

$G_q = (V_q, E_q)$: points vs. lines graph of $PG(2, q)$.

$$|V_q| = 2(q^2 + q + 1) \quad , \quad |E_q| = (q+1)(q^2 + q + 1).$$

Spectral Gap: $\mu_0(G_q) = 1 - \frac{\sqrt{q}}{q+1}$.

If $q \geq d^2$ then $\mu_0(G_q) > \frac{d-1}{d}$. This implies the following

Theorem [Garland]:

Let $q \geq d^2$ and let X be a pure d-dimensional complex such that $\mathrm{lk}(X, \tau) \cong G_q$ for all $\tau \in X(d-2)$.
Then $H_{d-1}(X; \mathbb{R}) = 0$.

Cohomology of Discrete Subgroups

$\mathbb{K}$ a local field with residue field $\mathbb{F}_q$.
Γ a torsion-free discrete cocompact subgroup of $SL_{d+1}(\mathbb{K})$.

Theorem [Garland]:

If $q \geq d^2$ then $H_{d-1}(\Gamma, \mathbb{R}) = 0$.

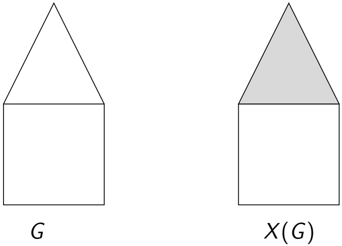
Sketch of Proof:

$\mathcal{B} = \tilde{A}_d(\mathbb{K})$ - the affine building associated to $SL_{d+1}(\mathbb{K})$.
$\mathcal{B}$ is a contractible complex with a free Γ action.
The quotient space $\mathrm{B}\Gamma = \mathcal{B}/\Gamma$ is a pure d-dimensional complex such that $\mathrm{lk}(\mathrm{B}\Gamma, \tau) \cong G_q$ for all $\tau \in \mathrm{B}\Gamma(d-2)$.
Therefore

$$H^{d-1}(\Gamma; \mathbb{R}) = H^{d-1}(\mathrm{B}\Gamma; \mathbb{R}) = 0.$$

Flag Complexes

The flag complex $X(G)$ of a graph $G = (V, E)$:
Vertex set: V, Simplices: all cliques σ of G.



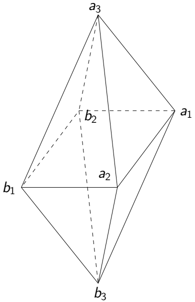
Remarks:

The first subdivision of a complex is a flag complex.

Face Numbers of Flag Complexes

Octahedral n-Sphere

$$(S^0)^{*(k+1)} =$$
$$\{a_1, b_1\} * \cdots * \{a_{k+1}, b_{k+1}\}$$



Proposition [M '03]:

If $\tilde{\mathrm{H}}_k(X(G)) \neq 0$ then for all j:

$$f_j(X(G)) \geq f_j((S^0)^{*(k+1)}) = \binom{k+1}{j+1} 2^{j+1}.$$

Homology of Flag Complexes of Random Graphs

Let $\epsilon > 0$ be fixed and let $G \in G(n, p)$.

Theorem [Kahle '12]:

$$p \leq n^{-\frac{1}{k}-\epsilon} \quad \Rightarrow \quad H_k(X(G); \mathbb{Z}) = 0 \ \text{ a.a.s.}$$

$$p \geq \left(\frac{(\frac{k}{2} + 1 + \epsilon) \log n}{n} \right)^{\frac{1}{k+1}} \quad \Rightarrow \quad H_k(X(G); \mathbb{R}) = 0 \ \text{ a.a.s.}$$

Theorem [DeMarco-Hamm-Kahn '12]:

$$p \geq \left(\frac{(\frac{3}{2} + \epsilon) \log n}{n} \right)^{\frac{1}{2}} \quad \Rightarrow \quad H_1(X(G); \mathbb{F}_2) = 0 \ \text{ a.a.s.}$$

Vanishing of $H_k(X(G);\mathbb{R})$

Let $C_k = \frac{k}{2} + 1 + \epsilon$ and let $p = \left(\frac{C_k \log n}{n}\right)^{\frac{1}{k+1}}$.

Claim 1: The $(k+1)$-skeleton $Y = X(G)^{(k+1)}$ is a.a.s. pure:

$$
\begin{aligned}
& E[\#\sigma \in Y(k) \text{ such that } \sigma \not\subset (k+1) - \text{face of } Y] \\
& = \binom{n}{k+1} p^{\binom{k+1}{2}} (1 - p^{k+1})^{n-k-1} \\
& \leq n^{k+1} \left(\frac{C_k \log n}{n}\right)^{\frac{k}{2}} \left(1 - \frac{C_k \log n}{n}\right)^{n-k-1} = O(n^{-\frac{\epsilon}{2}}).
\end{aligned}
$$

Claim 2: $\mu_0(\mathrm{lk}(Y,\tau)) = 1 - o(1)$ a.a.s. for all $\tau \in Y(k-1)$.

Claims 1 & 2 + Garland's Thm. $\Rightarrow$ $H_k(X(G);\mathbb{R}) = 0$ a.a.s.

Fundamental Group of $X(G)$

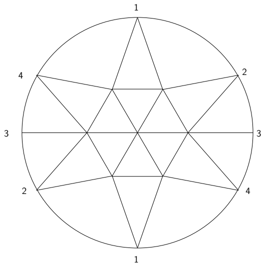
Theorem [Babson]:

If $n^{-\frac{1}{2}+\epsilon} < p < n^{-\frac{1}{3}-\epsilon}$ then a.a.s. $\pi_1(X(G))$ is nontrivial and hyperbolic.

Theorem [Costa-Farber-Horak]:

- If $n^{-\frac{11}{30}+\epsilon} < p < n^{-\frac{1}{3}-\epsilon}$ then a.a.s. $\pi_1(Y)$ has 2-torsion.
- If $n^{-\frac{1}{2}+\epsilon} < p < n^{-\frac{11}{30}-\epsilon}$ then a.a.s. $\pi_1(Y)$ is torsion free.

Flag Projective Plane



If $p = n^{-\frac{f_0}{f_1}+\epsilon} = n^{-\frac{11}{30}+\epsilon}$ then a.a.s. $X(G) \supset Z \cong \mathbb{P}^2$ such that $\mathbb{Z}_2 = \pi_1(Z) \hookrightarrow \pi_1(Y)$.

Spectral Gap and Homological Connectivity

Theorem [Aharoni-Berger-M]:

$G = (V, E)$ - a graph, $|V| = n$.

$$\lambda_2(G) > \frac{(k-1)n}{k} \quad \Rightarrow \quad \tilde{H}_i(X(G); \mathbb{R}) = 0 \quad \forall\, 0 \le i \le k-1.$$

The Bound is Sharp

$n = k\ell$.

$T_k(n)$ - the complete k-partite graph with equal sides.

$$\lambda_2(T_k(n)) = \frac{(k-1)n}{k} \quad , \quad \dim \tilde{H}_{k-1}(X(T_k(n)); \mathbb{R}) = (\ell - 1)^k \neq 0.$$

The theorem has applications to hypergraph matching ...

Bipartite Matching

$A_1, \ldots, A_m$ finite sets.
A System of Distinct Representatives (SDR):
a choice of distinct $x_1 \in A_1, \ldots, x_m \in A_m$.

A_1	A_2	A_3
1	1	
		2
3	3	3

$\exists$ SDR

A_1	A_2	A_3
1	1	
2		2

$\nexists$ SDR

Hall's Theorem (1935)

$(A_1, \ldots, A_m)$ has an SDR iff
$|\cup_{i \in \mathrm{I}} A_i| \geq |\mathrm{I}|$ for all $\mathrm{I} \subset [m] = \{1, \ldots, m\}$.

Hypergraph Matching

A Hypergraph is a family of sets $\mathcal{F} \subset 2^V$
$(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ a sequence of m hypergraphs
A System of Disjoint Representatives (SDR) for $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$
is a choice of pairwise disjoint $F_1 \in \mathcal{F}_1, \ldots, F_m \in \mathcal{F}_m$

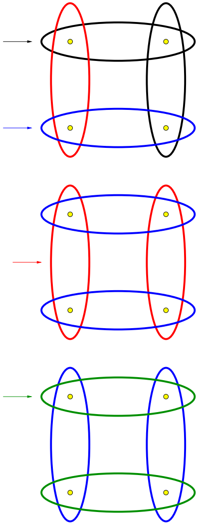
When do $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ have an SDR?

The problem is NP-Complete even if all $\mathcal{F}_i$'s consist of 2-element sets. Therefore, we cannot expect a "good" characterization as in Hall's Theorem.

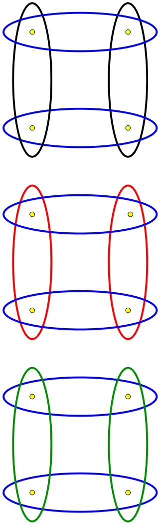
There are however some interesting sufficient conditions ...

Do $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)$ have an SDR?

$\exists$ SDR

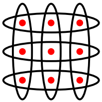


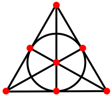
$\nexists$ SDR



The Aharoni-Haxell Theorem

A Matching is a hypergraph $\mathcal{M}$ of pairwise disjoint sets.
The Matching Number $\nu(\mathcal{F})$ of a hypergraph $\mathcal{F}$ is the maximal size $|\mathcal{M}|$ of a matching $\mathcal{M} \subset \mathcal{F}$.

$\nu(\mathcal{F}) = 3$ 

$\nu(\mathcal{F}) = 1$ 

The Aharoni-Haxell Theorem

$\mathcal{F}_1, \ldots, \mathcal{F}_m \subset \binom{V}{r}$ such that for all $\mathrm{I} \subset [m]$

$$\nu\Big(\bigcup_{i \in \mathrm{I}} \mathcal{F}_i\Big) > r(|\mathrm{I}| - 1) \quad .$$

Then $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ has an SDR.

A Fractional Extension

A Fractional Matching of a hypergraph $\mathcal{F}$ on V is a function $f : \mathcal{F} \to \mathbb{R}_+$ such that $\sum_{F \ni v} f(F) \leq 1$ for all $v \in V$.
The Fractional Matching Number $\nu^*(\mathcal{F})$ is $\max_f \sum_{F \in \mathcal{F}} f(F)$ over all fractional matchings f .

Example: The Finite Projective Plane $\mathcal{P}_n$

$\nu(\mathcal{P}_n) = 1 \quad , \quad \nu^*(\mathcal{P}_n) = \frac{n^2+n+1}{n+1}$

Theorem [ABM]:

$\mathcal{F}_1, \ldots, \mathcal{F}_m \subset \binom{V}{r}$ such that for all $\mathrm{I} \subset [m]$

$$\nu^*(\bigcup_{i \in \mathrm{I}} \mathcal{F}_i) > r(|\mathrm{I}| - 1) \quad .$$

Then $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ has an SDR.

Random Orientable Surfaces

Harer-Zagier Model

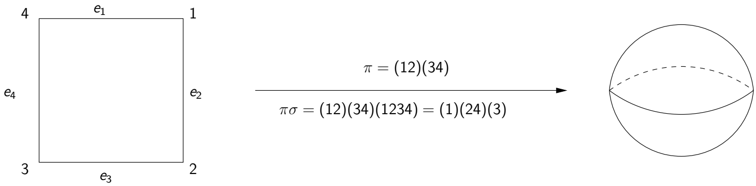
P - a regular $2n$-gon with vertices $1, \ldots, 2n$ and edges
$e_1 = (2n \to 1)$, $e_2 = (1 \to 2)$, ... , $e_{2n} = (2n - 1 \to 2n)$.
C_{2n} - the conjugacy class of fixed point free involutions in S_{2n}.
$|C_{2n}| = (2n-1)!! = (2n-1) \cdot (2n-3) \cdots 3 \cdot 1$.
For $\pi \in C_{2n}$ glue all pairs of edges $e_i, e_{\pi(i)}$ with reverse orientations to obtain an orientable surface $M(\pi)$.

Genus of $M(\pi)$

$N(\eta) =$ the number of cycles in $\eta \in S_{2n}$.
$\sigma = (123 \cdots 2n) \in S_{2n}$.
Fact: $f_0(M(\pi)) = N(\pi\sigma)$.
Therefore:

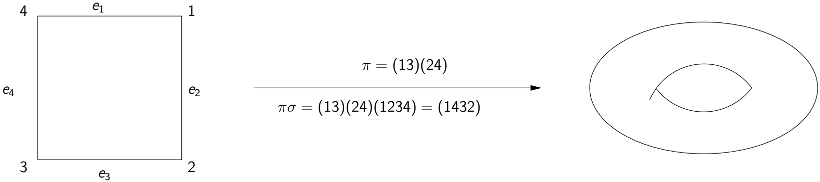
$$2 - 2 \cdot genus(M(\pi)) = f_0 - f_1 + f_2 = N(\pi\sigma) - n + 1.$$

Example: A Sphere



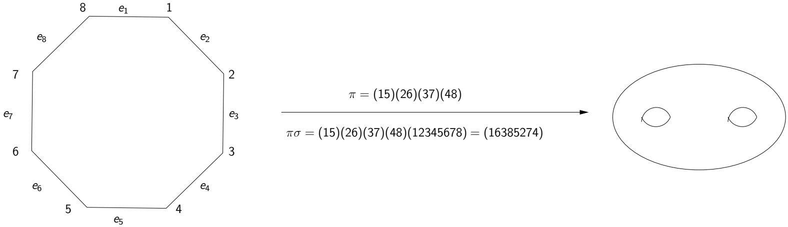
$$g = \frac{n + 1 - N(\pi\sigma)}{2} = \frac{2 + 1 - 3}{2} = 0.$$

Example: A Torus



$$g = \frac{n + 1 - N(\pi\sigma)}{2} = \frac{2 + 1 - 1}{2} = 1.$$

Example: A Genus 2 Surface



$$g = \frac{n + 1 - N(\pi\sigma)}{2} = \frac{4 + 1 - 1}{2} = 2.$$

Distribution of the Genus

Harer-Zagier Formula

Let $p_n(g) = \Pr[\ \pi \in C_{2n}\ :\ genus(M(\pi)) = g\ ]$.
Then:

$$1 + 2\sum_{n,g} p_n(g) k^{n+1-2g} x^{n+1} = \left(\frac{1+x}{1-x}\right)^k .$$

Expectation and Variance

$$E[genus] = \frac{1}{2}(n + 1 - \log(2n) - \gamma) + O(\frac{1}{n}).$$
$$Var[genus] = \frac{1}{4}(\log(2n) + \gamma - \frac{\pi^2}{6}) + O(\frac{\log n}{n}).$$